Engineering Notes

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Optimal Autorotational Descent of a Helicopter with Control and State Inequality Constraints

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Introduction

GLIDING descent in autorotation is used by helicopter pilots in case of engine failure. A successful landing following an autorotation descent requires considerable skill, and since it is seldom practiced, it is considered quite dangerous. Methods and devices have been proposed to improve helicopter autorotational landing characteristics. One passive concept is to store energy in the helicopter main rotor by using blades with high inertia.[1] However, the increased weight of the rotor blades reduces the helicopter payload, which is a disadvantage. Improved autorotation performance can also be achieved by optimal energy-management techniques.[2,3]

Johnson[2] used nonlinear optimal control theory to study the autorotational descent of a helicopter in hover. He found that the optimal descent from hover is purely vertical. A comparison of the results obtained with flight tests showed good correlation to verify the basic features of the model used.

Lee et al.[3] extended the work of Johnson by the addition of inequality constraints on the rotor thrust coefficient and helicopter sinkrate. Their study indicates that, subject to pilot acceptability, some reduction could be made in the H-V restriction zone using optimal control techniques. In this research, inequality constraints proposed in Ref. 3 are replaced by bounds on the rotor angular speed and collective pitch. The formulated problem is then solved using a zero-order optimization algorithm. Only results obtained for the most critical case of descents from an engine failure in hover are given.

Dynamic Performance Model of a Helicopter in Autorotation

We used the point-mass model of an OH-58A light single-rotor helicopter described in Refs. 2 and 3 in our study. This model includes effects due to the fuselage's parasite drag, the rotor's induced and profile power losses, and the induced velocity dynamics.[2,3] Table 1 of Ref. 3 summarized the values of the model parameters used.

The states of the model are the helicopter sinkrate V, rotor induced velocity ν, vertical displacement h (from the point at which engine failure occurred), and the rotor angular speed Ω. The control is the rotor thrust coefficient C_T, which is related to the collective pitch at 75% span, θ_{75} (Ref. 2):

$$\theta_{75} = \frac{6C_T}{a\sigma} + \frac{3}{2}\left(\frac{\nu - V}{\Omega R}\right) \tag{1}$$

where R, σ, and a are the rotor radius, solidity ratio, and rotor blade two-dimensional lift curve slope, respectively. A pilot reaction time of T_D s is included in our study.

Nonlinear Optimization Problem with Path Inequality Constraints

The helicopter is in hover at the time of engine failure with rotor speed Ω_0 and height h_0 above ground level. The optimization problem is to arrive at the ground with the least vertical velocity. The cost functional is

$$I = \tfrac{1}{2} x_{1f}^2 \tag{2}$$

Here $x_1 = V/0.01\Omega_0 R$ is the normalized vertical velocity. $(\,)_f$ denotes condition at the endtime. Other normalized quantities used are $\tau = \Omega_0 t/100$, $x_2 = \Omega/\Omega_0$, $x_3 = h/R$, $x_4 = \nu/0.01\Omega_0 R$, and $u_1 = 10^3 C_T$.

The rotor profile drag coefficient increases sharply when C_T/σ exceeds a stall limit $(C_T/\sigma)_{\rm stall}$.[3] To avoid this condition, Johnson[2] used a penalty term in the rotor's profile power-loss expression. Lee et al.[3] put hard bounds on C_T. In this research, we use bounds on the collective pitch angle θ_{75}:

$$0 \text{ deg} \le \theta_{75} \le 16 \text{ deg} \tag{3}$$

It is felt that these bounds are more closely related to the physical limits that exist on the pilot's collective stick position. In addition, state inequality constraints were used to limit the variations of the rotor angular speed. An upper bound on Ω will prevent the rotor from overspeeding, which would lead to unacceptable blade centrifugal stresses. In addition, the rotor must at all times be maintained above some level in order to control the helicopter. The assumed maximum and minimum rotor speeds were 1.15 Ω_0 and 0.50 Ω_0, respectively. Hence,

$$0.50 \le x_2 \le 1.15 \tag{4}$$

The optimization problem is therefore to minimize the cost functional I [cf. Eq. (2)] subject to[3]

$$x_1' = \tau_f(g_0 - m_0 x_2^2 u_1 - f_0 x_1^2) \tag{5}$$

$$x_2' = -\tau_f i_0 x_2^2 \Big[c_0(1 + q_0^2 u_1^2 + r_0^N u_1^N) + 0.01 u_1 (x_4 - x_1)/x_2\Big] \tag{6}$$

$$x_3' = \tau_f x_1 \tag{7}$$

$$x_4' = \tau_f t_0(-x_4 + 100 k_0 \sqrt{u_1} f_I f_G)\Big] \tag{8}$$

where $r_0^{-1} = (10^3 C_T)_{\rm stall} = 10^3 (C_T/\sigma)_{\rm stall} \times \sigma$. $(\,)'$ denotes differentiation with respect to ξ $(= \tau/\tau_f)$. g_0, m_0, and so forth are defined in Ref. 3. The initial condition is $(0,\ 1,\ 0,\ \sqrt{mg/2\rho\pi R^2}/0.01\Omega_0 R)^T$. The path inequality constraints are given by Eqs. (3) and (4), and the terminal constraint is $x_{3f} - h_0/R = 0$.

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Numerical Solution Techniques

The Nelder-Mead simplex method[4] is used in our work because it can effectively handle the state and control inequality constraints, initial pilot time delay, and the switching of the induced velocity between the vortex-ring and momentum theory states [2,3] formulated in the optimization problem. To use this method, the time history $C_T(\xi)$ is first expressed as a weighted sum of shifted Legendre polynomials[5] $P_k(\xi)$, where

$$P_k(\xi)=\sum_{j=0}^{k}\frac{(-1)^j(2k-j)!}{j![(k-j)!]^2}\,\xi^{k-j} \qquad (9)$$

and $C_T(\xi)=\Sigma_{i=0}^{i=7}\,p_iP_i(\xi)$. Here p_i, $i=0,1,\ldots,7$, are called the shifted Legendre coefficients. Only eight Legendre coefficients are used because higher-order polynomials cannot reduce the cost functional any further. These coefficients, as well as the unknown end-time τ_f, are the nine unknown parameters to be optimally determined by the simplex method.

In this method, a regular simplex is formed in the space of the unknown parameters. A cost function $\bar{J}$ is then evaluated at each vertex of the simplex. Here

$$\bar{J} = \tfrac{1}{2}x_{1f}^2 + W_0(x_{3f}-\bar{h}_f)^2 + \tfrac{1}{2}\int_0^1 W_1F_1(\xi)+W_2F_2(\xi)\,\mathrm{d}\xi \qquad (10)$$

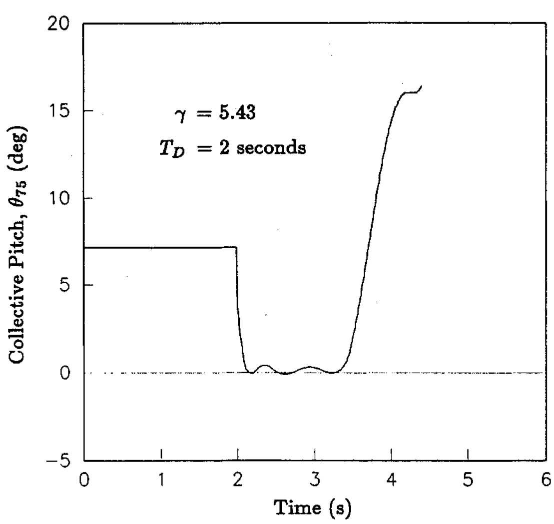

Fig. 1 Time history of the collective pitch (θ_{75}) from an entry height of 100 ft.

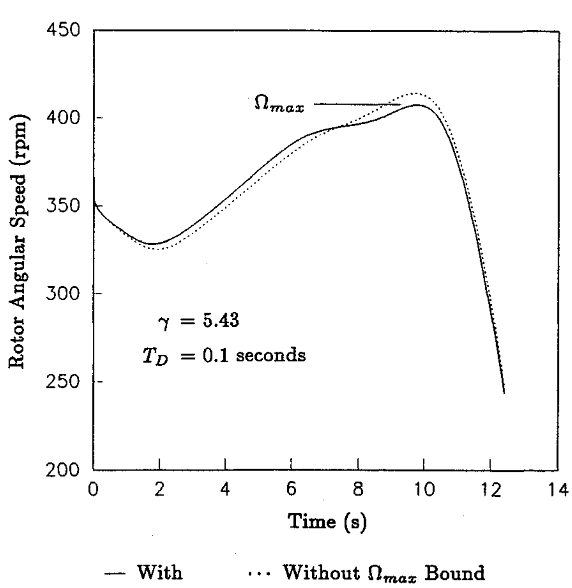

Fig. 2 Time histories of the rotor angular speed from an entry height of 500 ft.

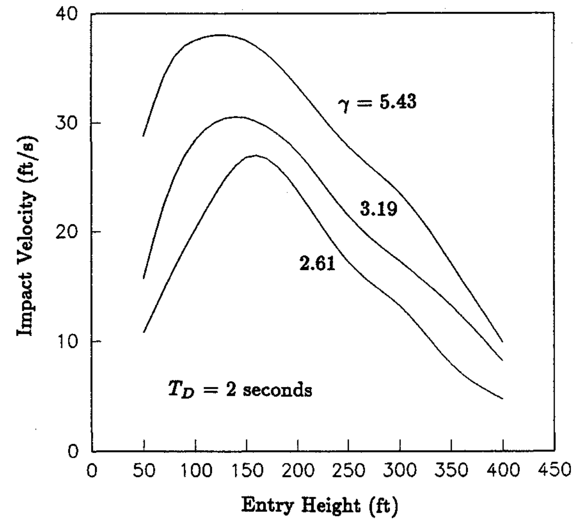

Fig. 3 Effects of entry height and rotor blade inertia on impact velocity.

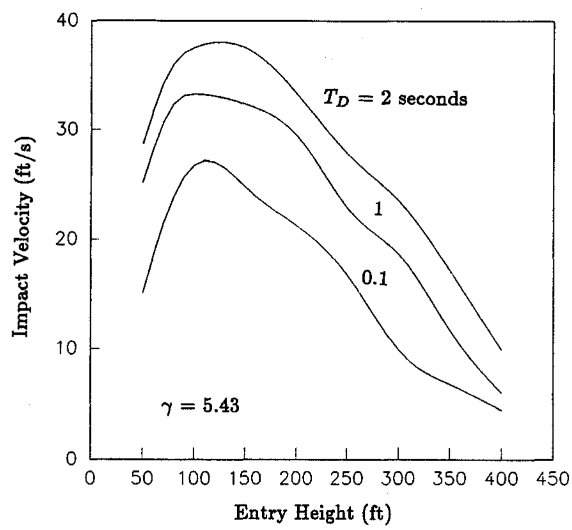

Fig. 4 Effects of entry height and pilot reaction time on impact velocity.

Here W_i, $i=0,\ldots,2$ are arbitrary large penalty weightings (e.g., $W_0=10$) used to enforce the path and terminal constraints of the formulated problem. The functions F_1 and F_2 are given by

$$F_1(\xi)=\begin{cases}0 & \text{if } S_1(\xi)\geq 0\\ S_1^2 & \text{otherwise}\end{cases} \qquad (11a)$$

$$F_2(\xi)=\begin{cases}0 & \text{if } S_2(\xi)\geq 0\\ S_2^2 & \text{otherwise}\end{cases} \qquad (11b)$$

where $S_1(\xi)=(\theta_{\max}-\theta_{75})(\theta_{75}-\theta_{\min})$ and $S_2(\xi)=(1.15-x_2)(x_2-0.50)$. When the values of $\bar{J}$ for all of the vertices are computed, the vertex with the highest value is located. The simplex then performs either a reflection, contraction, or expansion, and contracts itself to the final minimum. A detailed description of the technique is given in Ref. 4.

Optimal Control Results and Discussions

The nominal problem studied involves a helicopter with a lock number $\gamma=5.43$ and a 2-s pilot reaction time T_D. Rotors with larger blade inertias are used (with $T_D=2$ s) to study the effects of rotor blade inertia. Similarly, shorter pilot reaction times are used (with $\gamma=5.43$) to investigate the effects of pilot reaction time.

Figure 1 indicates that the collective pitch should be dropped to its lowest setting (0 deg) immediately after T_D s. This action reduces the induced power loss, and effectively arrests the high rate of rotor rpm loss. The collective pitch is then kept at 0 deg for 1.5 s before a collective flare is executed to generate lift to reduce the impact velocity. The collective pitch is on (or near) its upper bound during the last 0.5 s of travel. From this particular entry condition (with $h_0 = 100$ ft), neither the upper nor lower bound on the rotor rpm is violated.

The upper bound on the rotor rpm is violated when T_D is 0.1 s (with standard rotor blades) and the entry height is above 450 ft. Figure 2 shows the time histories of the rotor rpm obtained with and without a Ω_{max}-bound from an entry height of 500 ft. Without the Ω_{min}-bound, the peak rotor rpm is 415 rpm. The rotor will peak at an even higher rpm if the engine fails at a higher entry height. With a Ω_{max}-bound, the rotor rpm stays on the bound for a brief period of time, but never exceeded it. The Ω_{max}-bound is not active in this particular case.

In Fig. 3, the impact velocity is plotted against the entry height for three rotor lock numbers of 5.43, 3.19, and 2.61. For all of the cases studied, the impact velocity first increases, reaches a maximum, and then decreases with the entry height. For the cases studied, the critical entry height, i.e., the entry height with the maximum impact velocity, is between 100 and 150 ft. This critical height is close to the height of the "knee" of the low-speed H-V restriction zone. Figure 3 also indicates that the high-inertia rotor (with a smaller lock number) performs better than the low-inertia rotor for all of the entry heights studied.

Figure 4, which depicted results obtained for three pilot reaction times of 2, 1, and 0.1 s, indicates that the longer the pilot reaction time, the poorer the autorotational performance. For example, from an entry height of 150 ft, the impact velocity obtained with a 2-s reaction time is more than 50% higher than that found with a 0.1-s reaction time. This deterioration is due mainly to the severe bleeding of the rotor rpm, loss of vehicle altitude, as well as the buildup of vertical sinkrate within the pilot reaction time.

Concluding Remarks

A point-mass model of an OH-58A helicopter was used to determine its autorotation profiles that minimize the impact velocity of the helicopter while staying within bounds on the rotor's collective pitch and angular speed. The optimal control strategies obtained are similar to those used by pilots in autorotational landings. The study indicates that there is a potential for reducing the H-V restriction zone of OH-58A helicopters using optimal energy management techniques.

Acknowledgment

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How to Perform Differentiations on Matrices

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Introduction

In many linear system studies and formulations, a derivative of a matrix equation with respect to a matrix is desirable. Unfortunately, there is no easy way of performing such differentiation. This Note introduces a method of differentiating matrices easily through the definition of some unique operators and notations.

Definitions

Definition: Vec (·) Operator for Two-Dimensional Arrays[1]

For a two-dimensional array A of dimension $p \times q$

$$\mathrm{Vec}(A) \equiv \left[a_1^T a_2^T \cdots a_q^T\right]^T$$

where a_i is the ith column of array A. Note that the Vec operator "vectorizes" a two-dimensional array by stacking the columns together and reducing the dimension of the operand array by 1, i.e., from two dimensions to one dimension.

Example:

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\mathrm{Vec}(A) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}^T$$

Definition: Vec (·) Operator for Three-Dimensional Arrays

For a three-dimensional array B of dimension (m, n, p)

$$\mathrm{Vec}(B) \equiv \left[\mathrm{Vec}(B_{IJ,1}), \mathrm{Vec}(B_{IJ,2}) \cdots \mathrm{Vec}(B_{IJ,p})\right]$$

where $B_{IJ,k}$ is the kth subarray of B, where $1 \le k \le p$, $I = 1, \ldots, m$, $J = 1, \ldots, n$. Note that the Vec operator reduces the dimension of the operand array by 1, i.e., from three dimensions to two dimensions.

Example:

$$B(2,2,3) = \left[\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix} \begin{pmatrix} 9 & 11 \\ 10 & 12 \end{pmatrix}\right]$$

$$\mathrm{Vec}(B) = \left[\mathrm{Vec}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \mathrm{Vec}\begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix}, \mathrm{Vec}\begin{pmatrix} 9 & 11 \\ 10 & 12 \end{pmatrix}\right]$$

$$= \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}$$

Definition: Expanded Identity Matrix II

II is a three-dimensional array of dimension (x, y, xy), and Vec(II) is the identity matrix of dimension (xy, xy).

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